A classification of certain almost α-Kenmotsu manifolds

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Abstract. We study $\mathcal{D}$-homothetic deformations of almost α-Kenmotsu structures. We characterize almost contact metric manifolds which are CR-integrable almost α-Kenmotsu manifolds, through the existence of a canonical linear connection, invariant under $\mathcal{D}$-homothetic deformations. If the canonical connection associated to the structure (φ, ξ, η, g) has parallel torsion and curvature, then the local geometry is completely determined by the dimension of the manifold and the spectrum of the operator h' defined by $2\alpha h' = (\mathcal{L}_\xi \varphi) \circ \varphi$. In particular, the manifold is locally equivalent to a Lie group endowed with a left invariant almost α-Kenmotsu structure. In the case of almost α-Kenmotsu $(\kappa, \mu)'$-spaces, this classification gives rise to a scalar invariant depending on the real numbers κ and α.

2010 *Mathematics Subject Classification*: 53C15; 53C25.
Key words and phrases: almost α-Kenmotsu manifolds, $\mathcal{D}$-homothetic deformations, CR-manifolds, η-parallel structures, nullity distributions.

Introduction

Almost Kenmotsu manifolds are a special class of almost contact metric manifolds, recently investigated in [16, 14, 7, 8, 9]. An almost contact metric manifold $(M^{2n+1}, \varphi, \xi, \eta, g)$ is said to be an almost Kenmotsu manifold if $d\eta = 0$ and $d\Phi = 2\eta \wedge \Phi$, where Φ is the fundamental 2-form associated to the structure. Normal almost Kenmotsu manifolds are known as Kenmotsu manifolds [13]: they set up one of the three classes of almost contact metric manifolds whose automorphism group attains the maximum dimension [19].

The class of almost Kenmotsu manifolds is not invariant with respect to $\mathcal{D}$-

homothetic deformations, that is changes of the structure tensors of the form

$$\bar{\varphi} = \varphi, \qquad \bar{\xi} = \frac{1}{\beta}\xi, \qquad \bar{\eta} = \beta\eta, \qquad \bar{g} = \beta g + \beta(\beta - 1)\eta \otimes \eta, \tag{1}$$

where β is a positive constant. These deformations were introduced by Tanno in [18] and largely studied for the class of contact metric manifolds. Indeed, for an almost contact metric structure, such a change preserves the property of being contact metric, K-contact, Sasakian or strongly pseudo-convex CR, and the property for the characteristic vector field of a contact metric structure to belong to the (κ, μ)-nullity distribution. In [4] E. Boeckx provides a full classification of non-Sasakian contact metric (κ, μ)-spaces up to $\mathcal{D}$-homothetic deformations. He associates to each non-Sasakian (κ, μ)-space M an invariant I_M depending on the real numbers κ, μ, and provides an explicit example of such a space for every dimension $2n+1$ and for every value of the invariant.

In this paper we consider the class of almost α-Kenmotsu manifolds [16, 14, 12]. They are almost contact metric manifolds with structure (φ, ξ, η, g) such that $d\eta = 0$ and $d\Phi = 2\alpha\eta \wedge \Phi$, α being a non-zero real constant. Applying deformation (1), one obtains an almost $\frac{\alpha}{\beta}$-Kenmotsu structure.

After some preliminaries on general properties of almost α-Kenmotsu manifolds, dealing with the Levi-Civita connection and the Riemannian curvature, also under the hypothesis of local symmetry, we shall focus on some properties which are invariant under $\mathcal{D}$-homothetic deformations. The first one is the η-parallelism of the operator $h' = \frac{1}{2\alpha}(\mathcal{L}_\xi \varphi) \circ \varphi$, where $\mathcal{L}$ denotes the Lie derivative. The vanishing of the covariant derivative $\nabla_\xi h'$ is also an invariant property. If both these conditions are satisfied and $h' \neq 0$, then the spectrum of h' is of type $\{0, \lambda_1, -\lambda_1, \ldots, \lambda_r, -\lambda_r\}$, each λ_i being a positive constant. Denoting by $[0]$ the distribution of the eigenvectors of h' with eigenvalue 0 and orthogonal to ξ, and by $[\lambda_i]$ and $[-\lambda_i]$ the eigendistributions with eigenvalues λ_i and $-\lambda_i$ respectively, the manifold is locally the warped product

$$M' \times_{f_0} M_0 \times_{f_1} M_{\lambda_1} \times_{g_1} M_{-\lambda_1} \times_{f_2} \ldots \times_{f_r} M_{\lambda_r} \times_{g_r} M_{-\lambda_r},$$

where M' is an open interval, M_0, M_{λ_i} and $M_{-\lambda_i}$ are integral submanifolds of the distributions $[0]$, $[\lambda_i]$ and $[-\lambda_i]$. The warping functions are $f_0 = c_0 e^{\alpha t}$, $f_i = c_i e^{\alpha(1+\lambda_i)t}$ and $g_i = c'_i e^{\alpha(1-\lambda_i)t}$, with c_0, c_i and c'_i positive constants. Moreover, M_0 is an almost Kähler manifold and the structure is CR-integrable if and only if 0 is a simple eigenvalue or M_0 is a Kähler manifold (Theorem 4).

As a special case, we shall consider almost α-Kenmotsu manifolds whose characteristic vector field ξ belongs to the $(\kappa, \mu)'$-nullity distribution, that is, for some

real numbers κ, μ, the Riemannian curvature satisfies

$$R_{XY}\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)h'X - \eta(X)h'Y) \tag{2}$$

for all vector fields X and Y. Applying a $\mathcal{D}$-homothetic deformation, condition (2) is preserved up to a change of the real numbers κ, μ. We shall see that, for an almost α-Kenmotsu $(\kappa, \mu)'$-space, the operator h' is η-parallel and $\nabla_\xi h' = 0$. We also prove that $\kappa \leq -\alpha^2$. If $\kappa = -\alpha^2$, then $h' = 0$. If $\kappa < -\alpha^2$ then $\mu = -2\alpha^2$, the structure is CR-integrable and the Riemannian curvature is completely determined (Theorem 5).

In order to obtain a local classification of the above manifolds, up to $\mathcal{D}$-homothetic deformations, we consider in section 3 an invariant linear connection, called the canonical connection, which was introduced in [8] for almost Kenmotsu manifolds. The existence of this connection characterizes almost contact metric manifolds which are CR-integrable almost α-Kenmotsu manifolds; it can be viewed as the analogue of the Tanaka-Webster connection in contact geometry. In [5] E. Boeckx and J. T. Cho study Tanaka-Webster parallel spaces, i.e. CR-integrable contact metric manifolds for which the Tanaka-Webster connection has parallel torsion and curvature tensors; they prove that these spaces are Sasakian locally φ-symmetric spaces or non-Sasakian contact metric manifolds such that the characteristic vector field belongs to the $(\kappa, 2)$-nullity distribution.

Considering the canonical connection $\widetilde{\nabla}$ of a CR-integrable almost α-Kenmotsu manifold, we prove that the torsion $\widetilde{T}$ is parallel with respect to $\widetilde{\nabla}$ if and only if the tensor field h' is η-parallel and satisfies $\nabla_\xi h' = 0$. If, furthermore, the curvature tensor $\widetilde{R}$ satisfies $\widetilde{\nabla}\widetilde{R} = 0$, then $\widetilde{R}$ vanishes and this occurs if and only if 0 is a simple eigenvalue of h' or the integral submanifolds of the distribution $[\xi] \oplus [0]$ have constant Riemannian curvature $k = -\alpha^2$ (Theorem 7). For a fixed dimension of the manifold, supposing $\widetilde{\nabla}\widetilde{T} = 0$ and $\widetilde{\nabla}\widetilde{R} = 0$, we prove that the local geometry is completely determined, up to $\mathcal{D}$-homothetic deformations, by the spectrum of the operator h' (Theorem 9). In particular, the manifold is locally equivalent to a solvable non-nilpotent Lie group, which is a subgroup of the affine group $\mathit{Aff}(2n+1, \mathbb{R})$, endowed with a left invariant almost α-Kenmotsu structure, whose canonical connection coincides with the left invariant linear connection.

Applying the above classification to almost α-Kenmotsu $(\kappa, \mu)'$-spaces, with non-vanishing h', we obtain a scalar invariant I_M, depending on the real numbers κ and α. Together with the dimension of the manifold, I_M determines the local structure up to $\mathcal{D}$-homothetic deformations. We also show that such a manifold is locally $\mathcal{D}$-conformal to an almost cosymplectic manifold whose characteristic vector field ξ belongs to the κ_c-nullity distribution, with $\kappa_c = \kappa + \alpha^2$.

1 Preliminaries

An almost contact metric manifold is a differentiable manifold M^{2n+1} endowed with a structure (φ, ξ, η, g), given by a tensor field φ of type $(1,1)$, a vector field ξ, a 1-form η and a Riemannian metric g satisfying

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \varphi(\xi) = 0, \quad \eta \circ \varphi = 0,$$
$$g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y) \qquad \forall X, Y \in \mathfrak{X}(M).$$

Such a structure is said to be CR-integrable if the associated almost CR-structure $(\mathcal{D}, J)$ is integrable, where $\mathcal{D} = Im(\varphi) = Ker(\eta)$ is the $2n$-dimensional distribution orthogonal to ξ and J is the restriction of φ to $\mathcal{D}$. The structure is normal if the tensor field $N = [\varphi, \varphi] + 2d\eta \otimes \xi$ identically vanishes, where $[\varphi, \varphi]$ is the Nijenhuis torsion of φ. It is well known that normal almost contact metric manifolds are CR-manifolds [11]. We refer to [2, 3] for more details.

An almost α-Kenmotsu manifold is an almost contact metric manifold M^{2n+1} with structure (φ, ξ, η, g) such that

$$d\eta = 0, \qquad d\Phi = 2\alpha\eta \wedge \Phi, \tag{3}$$

where α is a non-zero real constant and Φ is the fundamental 2-form defined by $\Phi(X,Y) = g(X, \varphi Y)$ for any vector fields X and Y. Normal almost α-Kenmotsu manifolds are known as α-Kenmotsu manifolds. Let us consider the $(1,1)$-tensor field

$$h' := \frac{1}{2\alpha}(\mathcal{L}_\xi \varphi) \circ \varphi.$$

This operator satisfies $h'(\xi) = 0$, it is symmetric and anticommutes with φ. If X is an eigenvector of h' with eigenvalue λ, then φX is an eigenvector with eigenvalue $-\lambda$, and thus λ and $-\lambda$ have the same multiplicity. If $\lambda \neq 0$, we denote by $[\lambda]$ the distribution of the eigenvectors of h' with eigenvalue λ; if $\lambda = 0$, we denote by $[0]$ the distribution of the eigenvectors of h' with eigenvalue 0 and orthogonal to ξ, which has even rank.

The Levi-Civita connection of g satisfies $\nabla_\xi \varphi = 0$, which implies that $\nabla_\xi \xi = 0$ and $\nabla_\xi X \in \mathcal{D}$ for any $X \in \mathcal{D}$. Moreover, $\nabla_X \xi = \alpha(X + h'X - \eta(X)\xi)$ for any vector field X, or equivalently,

$$(\nabla_X \eta)(Y) = \alpha g(X + h'X, Y) - \alpha\eta(X)\eta(Y) \tag{4}$$

for all vector fields X, Y. From (3) it follows that the distribution $\mathcal{D}$ is integrable with almost Kähler leaves. The mean curvature vector field of the integral manifolds of $\mathcal{D}$ is $H = -\alpha\xi$ and these manifolds are totally umbilical if and only if $h' = 0$ [14].

An almost α-Kenmotsu structure is CR-integrable if and only if the tensor N vanishes on $\mathcal{D}$, or equivalently, the integral manifolds of $\mathcal{D}$ are Kähler manifolds. In terms of the Levi-Civita connection, the CR-integrability of the structure can be characterized by the condition

$$(\nabla_X \varphi)(Y) = \alpha g(\varphi X + \varphi h' X, Y)\xi - \alpha\eta(Y)(\varphi X + \varphi h' X) \tag{5}$$

for all vector fields X, Y, which is equivalent to the η-parallelism of the tensor field φ, that is $g((\nabla_X \varphi)Y, Z) = 0$ for any vector fields X, Y, Z orthogonal to ξ.

Analogously, the operator h' is said to be η-parallel if $g((\nabla_X h')Y, Z) = 0$ for every vector fields X, Y, Z orthogonal to ξ, and this is equivalent to requiring that

$$(\nabla_X h')Y = -\alpha g(Y, h'X + h'^2 X)\xi - \alpha\eta(Y)(h'X + h'^2 X) + \eta(X)(\nabla_\xi h')Y \tag{6}$$

for every vector fields X, Y.

Most of the results proved in [7] for the class of almost Kenmotsu manifolds can be generalized to the class of almost α-Kenmotsu manifolds. We omit the proofs since they are similar.

Theorem 1 *Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be an almost α-Kenmotsu manifold such that $h' = 0$. Then M^{2n+1} is locally a warped product $M' \times_f N^{2n}$, where N^{2n} is an almost Kähler manifold, M' is an open interval with coordinate t, $f = ce^{\alpha t}$, for some positive constant c.*

Proposition 1 *Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be an almost α-Kenmotsu manifold such that the integral manifolds of $\mathcal{D}$ are Kähler. Then, M^{2n+1} is an α-Kenmotsu manifold if and only if $h' = 0$, or equivalently, $\nabla\xi = -\alpha\varphi^2$. Therefore, a 3-dimensional almost α-Kenmotsu manifold such that $h' = 0$ is an α-Kenmotsu manifold.*

Consequently, an α-Kenmotsu manifold M^{2n+1} is a warped product of type $M' \times_f N^{2n}$, where M' is an open interval, N^{2n} is a Kähler manifold and $f = ce^{\alpha t}$, for some positive constant c.

As regards the Riemannian curvature of an almost α-Kenmotsu manifold, an easy computation shows that

$$R_{XY}\xi = \alpha^2(\eta(X)(Y + h'Y) - \eta(Y)(X + h'X)) + \alpha((\nabla_X h')Y - (\nabla_Y h')X) \tag{7}$$

for every vector fields X, Y, which implies that

$$R_{\xi X}\xi = \alpha^2(-\varphi^2 X + 2h'X + h'^2 X) + \alpha(\nabla_\xi h')X.$$

If the almost α-Kenmotsu manifold is locally symmetric, then the operator h' satisfies $\nabla_\xi h' = 0$, and for any unit eigenvector X of h' with eigenvalue λ, the ξ-sectional curvature is given by

$$K(\xi, X) = -\alpha^2(1+\lambda)^2,$$

which implies that $Ric(\xi,\xi) < 0$. The geometry of a locally symmetric almost α-Kenmotsu manifold is quite different in the two cases with vanishing or non-vanishing h'. Indeed, we have the following results.

Theorem 2 *Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be a locally symmetric almost α-Kenmotsu manifold. Then, M^{2n+1} is an α-Kenmotsu manifold if and only if $h' = 0$; in this case the manifold has constant sectional curvature $k = -\alpha^2$.*

Given an almost α-Kenmotsu manifold of constant curvature k, it can be proved that $h' = 0$, and the above Theorem implies that the structure is normal and $k = -\alpha^2$. In the case of non-vanishing h' we have

Theorem 3 *Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be a locally symmetric almost α-Kenmotsu manifold with $h' \neq 0$. Then the operator h' admits the eigenvalues $+1$ and -1. If, moreover, the Riemannian curvature satisfies $R_{XY}\xi = 0$ for any $X, Y \in \mathcal{D}$, then the spectrum of h' is $\{0, 1, -1\}$, with 0 as simple eigenvalue. The distributions $[\xi] \oplus [+1]$ and $[-1]$ are integrable with totally geodesic leaves and M^{2n+1} is locally isometric to the Riemannian product of an $(n+1)$-dimensional manifold of constant curvature $-4\alpha^2$ and a flat n-dimensional manifold.*

In the following we consider almost α-Kenmotsu manifolds with $\alpha > 0$. Notice that if (φ, ξ, η, g) is an almost α-Kenmotsu structure with $\alpha < 0$, then $(\varphi, -\xi, -\eta, g)$ is an almost α'-Kenmotsu structure with $\alpha' = -\alpha > 0$.

2 $\mathcal{D}$-homothetic deformations

Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be an almost α-Kenmotsu manifold and $(\bar\varphi, \bar\xi, \bar\eta, \bar g)$ the almost $\frac{\alpha}{\beta}$-Kenmotsu structure obtained by the $\mathcal{D}$-homothetic deformation (1). Notice that the operators h' and $\bar h'$ associated to these structures coincide. Let ∇ and $\bar\nabla$ be the Levi-Civita connections of g and $\bar g$ respectively. We prove that for all vector fields X, Y,

$$\bar\nabla_X Y = \nabla_X Y + \alpha\frac{\beta-1}{\beta}(g(X + h'X, Y) - \eta(X)\eta(Y))\xi. \tag{8}$$

Indeed, applying the Koszul formula and $d\eta = 0$, we have

$$\bar{\nabla}_X Y = \nabla_X Y + \frac{\beta - 1}{\beta}(\nabla_X \eta)(Y)\xi$$

and using (4), we obtain (8). The covariant derivatives of φ and h' satisfy

$$(\bar{\nabla}_X \varphi)(Y) = (\nabla_X \varphi)(Y) + \alpha\frac{\beta - 1}{\beta} g(X + h'X, \varphi Y)\xi,$$

$$(\bar{\nabla}_X h')(Y) = (\nabla_X h')(Y) + \alpha\frac{\beta - 1}{\beta} g(X + h'X, h'Y)\xi,$$

for all vector fields X and Y, so that the property for the tensor fields φ and h' to be η-parallel and the vanishing of the covariant derivative $\nabla_\xi h'$ are invariant under $\mathcal{D}$-homothetic deformations.

An easy computation shows that the Riemannian curvature tensors R and $\bar{R}$ of g and $\bar{g}$ are related by the following formula:

$$\begin{aligned} \bar{R}_{XY}Z = R_{XY}Z &+ \alpha\frac{\beta - 1}{\beta} g((\nabla_X h')Y - (\nabla_Y h')X, Z)\xi \\ &+ \alpha^2\frac{\beta - 1}{\beta}(g(Y + h'Y, Z) - \eta(Y)\eta(Z))(X + h'X) \\ &- \alpha^2\frac{\beta - 1}{\beta}(g(X + h'X, Z) - \eta(X)\eta(Z))(Y + h'Y) \end{aligned} \tag{9}$$

for every vector fields X, Y, Z. It follows that $\bar{R}_{XY}\xi = R_{XY}\xi$ for every vector fields X, Y. If ξ belongs to the $(\kappa, \mu)'$-nullity distribution, i.e. the Riemannian curvature tensor satisfies (2), then $\bar{\xi}$ belongs to the $(\bar{\kappa}, \bar{\mu})'$-nullity distribution, with

$$\bar{\kappa} = \frac{\kappa}{\beta^2}, \qquad \bar{\mu} = \frac{\mu}{\beta^2}.$$

Let us analyze now the geometry of almost α-Kenmotsu manifolds such that h' is η-parallel and satisfies $\nabla_\xi h' = 0$.

Theorem 4 *Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be an almost α-Kenmotsu manifold such that h' is η-parallel and $\nabla_\xi h' = 0$. Then the eigenvalues of the operator h' are constant. Let $\{0, \lambda_1, -\lambda_1, \ldots, \lambda_r, -\lambda_r\}$ be the spectrum of h', with $\lambda_i > 0$. Then M^{2n+1} is locally the warped product*

$$M' \times_{f_0} M_0 \times_{f_1} M_{\lambda_1} \times_{g_1} M_{-\lambda_1} \times_{f_2} \ldots \times_{f_r} M_{\lambda_r} \times_{g_r} M_{-\lambda_r}, \tag{10}$$

where M' is an open interval, M_0, M_{λ_i} and $M_{-\lambda_i}$ are integral submanifolds of the distributions $[0]$, $[\lambda_i]$ and $[-\lambda_i]$ respectively. The warping functions are $f_0 = c_0 e^{\alpha t}$, $f_i = c_i e^{\alpha(1+\lambda_i)t}$ and $g_i = c'_i e^{\alpha(1-\lambda_i)t}$, with c_0, c_i and c'_i positive constants. Finally, M_0 is an almost Kähler manifold and the structure is CR-integrable if and only if 0 is a simple eigenvalue or M_0 is a Kähler manifold.

Proof. The result is proved in [9] for almost Kenmotsu manifolds, corresponding to the case $\alpha = 1$. Let us consider an almost α-Kenmotsu structure (φ, ξ, η, g), with $\alpha \neq 1$, such that h' is η-parallel and $\nabla_\xi h' = 0$. Applying the $\mathcal{D}$-homothetic deformation (1) with $\beta = \alpha$, we obtain an almost Kenmotsu structure $(\bar\varphi, \bar\xi, \bar\eta, \bar g)$ such that $\bar h'$ is $\bar\eta$-parallel and $\bar\nabla_{\bar\xi} \bar h' = 0$, and the result applies to this structure. In particular, the distributions $[0]$, $[\lambda_i]$ and $[-\lambda_i]$ are integrable and for any distinct eigenvalues $\lambda_{i_1}, \dots, \lambda_{i_s}$ of h', the distribution $[\xi] \oplus [\lambda_{i_1}] \oplus \cdots \oplus [\lambda_{i_s}]$ is integrable with totally geodesic leaves with respect to $\bar g$; (8) implies that such leaves are totally geodesic also with respect to g.

Let us consider an eigenvalue $\lambda \neq 0$ of h'. We prove that the leaves of the distribution $[\lambda]$ are totally umbilical. Indeed, since $[\xi] \oplus [\lambda]$ is totally geodesic, choosing a local orthonormal frame $\{e_i\}$ of $[\lambda]$, the second fundamental form satisfies $I\!I(e_i, e_j) = -\alpha(1+\lambda)\delta_{ij}\xi$; the mean curvature vector field is $H = -\alpha(1+\lambda)\xi$ and, for any $X, Y \in [\lambda]$, we have $I\!I(X,Y) = g(X,Y)H$, so that the leaves of $[\lambda]$ are totally umbilical. Since the orthogonal distribution $[\lambda]^\perp$ is integrable with totally geodesic leaves, then M^{2n+1} is locally a warped product $B \times_{f_\lambda} M_\lambda$ such that $TB = [\lambda]^\perp$ and $TM_\lambda = [\lambda]$ (see [10]). We denote by g_0 and $\hat g$ the Riemannian metrics on B and M_λ respectively, such that the warped metric is given by $g_0 + f_\lambda^2 \hat g$. The projection $\pi : B \times_{f_\lambda} M_\lambda \to B$ is a Riemannian submersion with horizontal distribution $\mathcal{H} = [\lambda]^\perp$ and vertical distribution $\mathcal{V} = [\lambda]$. The mean curvature vector field $H = -\alpha(1+\lambda)\xi$ of the immersed submanifold $(M_\lambda, \hat g)$ is π-related to $-\frac{1}{f_\lambda} \operatorname{grad}_{g_0} f_\lambda$ ([1], 9.104) and thus, $\alpha(1+\lambda) f_\lambda \xi = \operatorname{grad}_{g_0} f_\lambda$. If m_λ is the multiplicity of λ, we choose local coordinates $\{t, x^1, \dots, x^{2n-m_\lambda}\}$ on B such that $\xi = \frac{\partial}{\partial t}$ and $\frac{\partial}{\partial x^i} \in [\lambda]$ for any $i = 1, \dots, 2n - m_\lambda$. Hence, we get $f_\lambda = c_\lambda e^{\alpha(1+\lambda)t}$, $c_\lambda > 0$.

Now, let us consider $TB = [\xi] \oplus [-\lambda] \oplus \bigoplus_{\mu \neq \pm\lambda} [\mu]$. The distribution $[\xi] \oplus \bigoplus_{\mu \neq \pm\lambda} [\mu]$ is integrable with totally geodesic leaves in M^{2n+1} and $[-\lambda]$ is integrable with totally umbilical leaves in M^{2n+1}. Since B is a totally geodesic submanifold of M^{2n+1}, these distributions are respectively totally geodesic and totally umbilical in B and, arguing as above, B is locally a warped product. This argument can be applied to each distribution $[\lambda_i]$ and $[-\lambda_i]$, $i \in \{1, \dots, r\}$, obtaining that M^{2n+1} is locally the warped product

$$N \times_{f_1} M_{\lambda_1} \times_{g_1} M_{-\lambda_1} \times_{f_2} \dots \times_{f_r} M_{\lambda_r} \times_{g_r} M_{-\lambda_r},$$

where $f_i = c_i e^{\alpha(1+\lambda_i)t}$ and $g_i = c_i' e^{\alpha(1-\lambda_i)t}$, with c_i and c_i' positive constants. The manifold N is a totally geodesic submanifold of M^{2n+1} and it is an integral submanifold of the distribution $[\xi] \oplus [0]$. By Theorem 1, N is locally a warped product $M' \times_{f_0} M_0$ of an open interval M' and an almost Kähler manifold M_0, with $f_0 = c_0 e^{\alpha t}$, $c_0 > 0$. $\square$

Under the hypotheses of the above Theorem, applying (6), we have

$$(\nabla_X h')Y - (\nabla_Y h')X = -\alpha\eta(Y)(h'X + h'^2 X) + \alpha\eta(X)(h'Y + h'^2 Y) \tag{11}$$

for any $X, Y \in \mathfrak{X}(M)$. Now, if we suppose that $Sp(h') = \{0, \lambda, -\lambda\}$, with 0 simple eigenvalue, then $h'^2 = \lambda^2(I - \eta \otimes \xi)$ and thus, from (11) and (7) it follows that

$$R_{XY}\xi = -\alpha^2(1+\lambda^2)(\eta(Y)X - \eta(X)Y) - 2\alpha^2(\eta(Y)h'X - \eta(X)h'Y).$$

Hence, we have

Proposition 2 *Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be an almost α-Kenmotsu manifold such that h' is η-parallel and $\nabla_\xi h' = 0$. If $Sp(h') = \{0, \lambda, -\lambda\}$, with 0 simple eigenvalue, then ξ belongs to the $(\kappa, \mu)'$-nullity distribution, with $\kappa = -\alpha^2(1+\lambda^2)$ and $\mu = -2\alpha^2$.*

As regards almost α-Kenmotsu $(\kappa, \mu)'$-spaces we have the following result.

Theorem 5 *Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be an almost α-Kenmotsu manifold such that ξ belongs to the $(\kappa, \mu)'$-nullity distribution. Then $\kappa \leq -\alpha^2$.*

If $\kappa = -\alpha^2$, then $h' = 0$ and M^{2n+1} is locally a warped product $M' \times_f N^{2n}$, where N^{2n} is an almost Kähler manifold, M' is an open interval with coordinate t, $f = ce^{\alpha t}$, for some positive constant c.

If $\kappa < -\alpha^2$, then $h' \neq 0$, $\mu = -2\alpha^2$ and $Spec(h') = \{0, \lambda, -\lambda\}$, with 0 as simple eigenvalue and $\lambda = \sqrt{-1 - \frac{\kappa}{\alpha^2}}$. The operator h' is η-parallel and satisfies $\nabla_\xi h' = 0$. The integral manifolds of $\mathcal{D}$ are Kähler manifolds. The distributions $[\lambda]$ and $[-\lambda]$ are integrable with totally umbilical leaves; the distributions $[\xi] \oplus [\lambda]$ and $[\xi] \oplus [-\lambda]$ are integrable with totally geodesic leaves. Finally, M^{2n+1} is locally isometric to the warped products

$$B^{n+1}(\kappa + 2\alpha^2\lambda) \times_f \mathbb{R}^n, \qquad \mathbb{H}^{n+1}(\kappa - 2\alpha^2\lambda) \times_{f'} \mathbb{R}^n,$$

where $B^{n+1}(\kappa + 2\alpha^2\lambda)$ is a space of constant curvature $\kappa + 2\alpha^2\lambda \leq 0$, tangent to the distribution $[\xi] \oplus [-\lambda]$, $\mathbb{H}^{n+1}(\kappa - 2\alpha^2\lambda)$ is the hyperbolic space of constant curvature $\kappa - 2\alpha^2\lambda < -\alpha^2$, tangent to the distribution $[\xi] \oplus [\lambda]$, $f = ce^{\alpha(1+\lambda)t}$ and $f' = c'e^{\alpha(1-\lambda)t}$, with c, c' positive constants.

Proof. The result is proved in [8] for almost Kenmotsu manifolds. Let us consider an almost α-Kenmotsu structure (φ, ξ, η, g), with $\alpha \neq 1$, such that ξ belongs to the $(\kappa, \mu)'$-nullity distribution. Applying the $\mathcal{D}$-homothetic deformation (1) with $\beta = \alpha$, we obtain an almost Kenmotsu structure $(\bar\varphi, \bar\xi, \bar\eta, \bar g)$ such that $\bar\xi$ belongs to the $(\bar\kappa, \bar\mu)'$-nullity distribution, with $\bar\kappa = \frac{\kappa}{\alpha^2}$, and $\bar\mu = \frac{\mu}{\alpha^2}$. Then $\bar\kappa \leq -1$. If $\bar\kappa = -1$, or equivalently $\kappa = -\alpha^2$, then $h' = 0$ and we apply Theorem 1.

If $\bar\kappa < -1$ then $\bar h' \neq 0$, $\bar\mu = -2$ and $Spec(\bar h') = \{0, \lambda, -\lambda\}$, with 0 as simple eigenvalue and $\lambda = \sqrt{-1-\bar\kappa}$. The tensor fields φ and h' are η-parallel and $\nabla_\xi h' = 0$, since these properties are invariant under $\mathcal{D}$-homothetic deformations; in particular, the integral manifolds of $\mathcal{D}$ are Kähler manifolds. From Theorem 4 it follows that M^{2n+1} is locally the warped product

$$M' \times_f M_\lambda \times_{f'} M_{-\lambda},$$

where M' is an open interval, M_λ and $M_{-\lambda}$ are integral submanifolds of the distributions $[\lambda]$ and $[-\lambda]$ respectively, $f = ce^{\alpha(1+\lambda)t}$ and $f' = c'e^{\alpha(1-\lambda)t}$, with c, c' positive constants.

We compute now the Riemannian curvature of M^{2n+1}. Recall that the integral submanifolds of the distribution $[\xi] \oplus [\lambda]$ have constant Riemannian curvature $\bar\kappa - 2\lambda$ with respect to the deformed Riemannian metric $\bar g$. Let us compute the relation between the curvature tensors R and $\bar R$ of g and $\bar g$ respectively. Combining (7) with the $(\kappa, \mu)'$-nullity condition, $\mu = -2\alpha^2$, we get

$$\alpha((\nabla_X h')Y - (\nabla_Y h')X) = (\kappa + \alpha^2)(\eta(Y)X - \eta(X)Y) - \alpha^2(\eta(Y)h'X - \eta(X)h'Y),$$

and thus, applying (9), we obtain

$$\begin{aligned}\bar R_{XY}Z = R_{XY}Z &+ \alpha(\alpha-1)(\eta(Y)g(X - h'X, Z) - \eta(X)g(Y - h'Y, Z))\xi \\ &+ \kappa\frac{\alpha-1}{\alpha}(\eta(Y)g(X,Z) - \eta(X)g(Y,Z))\xi \\ &+ \alpha(\alpha-1)(g(Y + h'Y, Z) - \eta(Y)\eta(Z))(X + h'X) \\ &- \alpha(\alpha-1)(g(X + h'X, Z) - \eta(X)\eta(Z))(Y + h'Y)\end{aligned}$$

for any $X, Y, Z \in \mathfrak{X}(M)$. On the distribution $[\xi] \oplus [\lambda]$ we have $h' = \lambda(I - \eta \otimes \xi)$ and applying the above formula, for any $X, Y, Z \in [\xi] \oplus [\lambda]$, we get

$$R_{XY}Z = -\alpha^2(1+\lambda)^2(g(Y,Z)X - g(X,Z)Y).$$

Therefore, the leaves of the distribution $[\xi] \oplus [\lambda]$ have constant Riemannian curvature $-\alpha^2(1+\lambda)^2 = \kappa - 2\alpha^2\lambda < -\alpha^2$ with respect to g and analogously, the leaves of the

distribution $[\xi]\oplus[-\lambda]$ have constant Riemannian curvature $-\alpha^2(1-\lambda)^2 = \kappa+2\alpha^2\lambda \leq 0$. Then, M^{2n+1} is locally isometric to the warped products

$$\mathbb{H}^{n+1}(\kappa - 2\alpha^2\lambda) \times_{f'} M_{-\lambda}, \qquad B^{n+1}(\kappa + 2\alpha^2\lambda) \times_f M_\lambda.$$

We prove that the fibers of the two warped products are flat Riemannian spaces. Denote by g_0 and $\hat{g}$ the Riemannian metrics on $\mathbb{H}^{n+1}(\kappa-2\alpha^2\lambda)$ and M_λ respectively, such that the first warped metric is given by $g_0 + f'^2\hat{g}$. Applying Proposition 7.42 in [17], for any $U, V, W \in [-\lambda]$, we have

$$\hat{R}_{UV}W = R_{UV}W - \frac{\|\mathrm{grad} f'\|^2}{f'^2}(g(U,W)V - g(V,W)U).$$

On the other hand, $R_{UV}W = -\alpha^2(1-\lambda)^2(g(V,W)U - g(U,W)V)$ and $\|\mathrm{grad} f'\|^2 = \alpha^2(1-\lambda)^2 f'^2$. Then, $\hat{R}_{UV}W = 0$. Analogously, the fibers of the second warped product are flat Riemannian spaces. $\square$

Under the hypotheses of the above Theorem, if $\lambda = 1$ then both the distributions $[\xi]\oplus[+1]$ and $[-1]$ are integrable with totally geodesic leaves and the manifold turns out to be locally isometric to the Riemannian product $\mathbb{H}^{n+1}(-4\alpha^2) \times \mathbb{R}^n$, which is locally symmetric. Conversely, supposing that M^{2n+1} is locally symmetric, then, by Theorem 3, $\lambda = 1$ and M^{2n+1} is locally isometric to $\mathbb{H}^{n+1}(-4\alpha^2) \times \mathbb{R}^n$. Hence, we have

Corollary 1 *Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be an almost α-Kenmotsu manifold such that $h' \neq 0$ and ξ belongs to the $(\kappa,\mu)'$-nullity distribution, $\mu = -2\alpha^2$. Then M^{2n+1} is locally symmetric if and only if Spec$(h') = \{0, 1, -1\}$, or equivalently $\kappa = -2\alpha^2$, in which case the manifold is locally isometric to $\mathbb{H}^{n+1}(-4\alpha^2) \times \mathbb{R}^n$.*

As another consequence of Theorem 5, we can obtain more information on the Riemannian curvature of an almost α-Kenmotsu manifold $(M^{2n+1}, \varphi, \xi, \eta, g)$ such that h' is η-parallel and $\nabla_\xi h' = 0$, as in the hypotheses of Theorem 4. Indeed, for any eigenvalue λ of the operator h', the distribution $[\xi]\oplus[\lambda]\oplus[-\lambda]$ is integrable with totally geodesic leaves which inherit an almost α-Kenmotsu structure from M^{2n+1}. If $\lambda = 0$, then the distribution $[\xi] \oplus [\lambda] \oplus [-\lambda]$ reduces to $[\xi] \oplus [0]$ and the leaves are local warped products $M' \times_{f_0} M_0$, where M_0 is a Kähler manifold in hypothesis of CR-integrability. If $\lambda > 0$ then, by Proposition 2, the leaves of $[\xi] \oplus [\lambda] \oplus [-\lambda]$ are almost α-Kenmotsu manifolds with characteristic vector field belonging to the $(\kappa,\mu)'$-nullity distribution, with $\kappa = -\alpha^2(1+\lambda^2)$ and $\mu = -2\alpha^2$. By Theorem 5, the leaves of $[\xi] \oplus [\lambda]$ have constant Riemannian curvature $\kappa - 2\alpha^2\lambda$ and the leaves of $[\xi] \oplus [-\lambda]$ have constant Riemannian curvature $\kappa + 2\alpha^2\lambda$.

3 The canonical connection

Theorem 6 *Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be an almost contact metric manifold. Then M^{2n+1} is a CR-integrable almost α-Kenmotsu manifold if and only if there exists a linear connection $\widetilde{\nabla}$ such that the tensor fields φ, g, η are parallel with respect to $\widetilde{\nabla}$ and the torsion $\widetilde{T}$ satisfies:*

a) $\widetilde{T}(X,Y) = 0$, *for any* $X, Y \in \mathcal{D}$,

b) $2\widetilde{T}(\xi, X) = \alpha(X + h'X)$, *for any* $X \in \mathcal{D}$,

c) $\widetilde{T}_\xi$ *is selfadjoint.*

The connection $\widetilde{\nabla}$ is invariant under $\mathcal{D}$-homothetic deformations and it is uniquely determined by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \alpha g(X + h'X, Y)\xi - \alpha\eta(Y)(X + h'X), \tag{12}$$

where ∇ is the Levi-Civita connection. The connection $\widetilde{\nabla}$ will be called the canonical connection associated to the structure (φ, ξ, η, g).

Proof. The result of existence and uniqueness of the connection is proved in [8] for almost Kenmotsu manifolds. Let $(\bar{\varphi}, \bar{\xi}, \bar{\eta}, \bar{g})$ be the almost contact metric structure obtained from (φ, ξ, η, g) through deformation (1) with $\beta = \alpha$. Then (φ, ξ, η, g) is a CR-integrable almost α-Kenmotsu structure if and only if $(\bar{\varphi}, \bar{\xi}, \bar{\eta}, \bar{g})$ is a CR-integrable almost Kenmotsu structure, and this is equivalent to the existence of a unique linear connection $\widetilde{\nabla}$ such that the tensor fields $\bar{\varphi}$, $\bar{g}$ and $\bar{\eta}$ are parallel with respect to $\widetilde{\nabla}$, and the torsion $\widetilde{T}$ vanishes on $\mathcal{D}$ and satisfies

b′) $2\widetilde{T}(\bar{\xi}, X) = X + \bar{h}'X$, for any $X \in \mathcal{D}$,

c′) $\widetilde{T}_{\bar{\xi}}$ is selfadjoint with respect to $\bar{g}$.

The parallelism of $\bar{\varphi}$, $\bar{g}$, $\bar{\eta}$ is equivalent to the parallelism of φ, g, η and b′) is obviously equivalent to b). Moreover, for any vector fields X, Y, we have

$$\bar{g}(\widetilde{T}_{\bar{\xi}} X, Y) = g(\widetilde{T}_\xi X, Y) + (\alpha - 1)\bar{\eta}(\widetilde{T}_{\bar{\xi}} X)\eta(Y).$$

If $\widetilde{T}_{\bar{\xi}}$ is selfadjoint with respect to $\bar{g}$, then $\widetilde{T}_\xi$ is selfadjoint with respect to g since $\bar{\eta}(\widetilde{T}_{\bar{\xi}} X) = \bar{g}(X, \widetilde{T}_{\bar{\xi}} \bar{\xi}) = 0$. Hence, c′) implies c). Analogously, one verifies that c) implies c′).

Denoting by $\bar{\nabla}$ the Levi-Civita connection of $\bar{g}$, for any vector fields X and Y, we have

$$\widetilde{\nabla}_X Y = \bar{\nabla}_X Y + \bar{g}(X + \bar{h}'X, Y)\bar{\xi} - \bar{\eta}(Y)(X + \bar{h}'X),$$

and applying (8) with $\beta = \alpha$, we get (12). Finally, the connection is invariant under $\mathcal{D}$-homothetic deformations. Indeed, if $\widetilde{\nabla}$ is the canonical connection associated to the almost α-Kenmotsu structure (φ, ξ, η, g), it can be easily verified that $\widetilde{\nabla}$ satisfies the axioms defining the canonical connection associated to the almost $\frac{\alpha}{\beta}$-Kenmotsu structure $(\bar{\varphi}, \bar{\xi}, \bar{\eta}, \bar{g})$ obtained through a $\mathcal{D}$-homothetic deformation of constant β. □

Now, let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be a CR-integrable almost α-Kenmotsu manifold. Let $\widetilde{\nabla}$ be the canonical connection and $\widetilde{R}$ its curvature tensor. A straightforward computation using (12) shows that for every vector fields X, Y, Z

$$\begin{aligned}\widetilde{R}_{XY}Z = R_{XY}Z &+ \alpha^2(g(Y + h'Y, Z)(X + h'X) - g(X + h'X, Z)(Y + h'Y)) \\ &+ \alpha g((\nabla_X h')Y - (\nabla_Y h')X, Z)\xi - \alpha\eta(Z)((\nabla_X h')Y - (\nabla_Y h')X),\end{aligned} \tag{13}$$

where R is the Riemannian curvature tensor. Consequently, we obtain the following result.

Proposition 3 *Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be an α-Kenmotsu manifold. Then the following conditions are equivalent:*

a) $\widetilde{\nabla}\widetilde{R} = 0$,

b) $\widetilde{R} = 0$,

c) *M^{2n+1} has constant Riemannian curvature $k = -\alpha^2$,*

d) *M^{2n+1} is a locally symmetric Riemannian manifold.*

Proof. Since the structure is normal, the operator h' vanishes and the equivalence of b) and c) immediately follows from (13). In order to prove the equivalence of a) and b), we show that for any $X \in \mathcal{D}$, $\widetilde{\nabla}_\xi X = \alpha X$. Indeed, the manifold is locally a warped product of an open interval M', which is tangent to the vector field ξ, and a Kähler manifold N^{2n}, orthogonal to ξ. Therefore, $[\xi, X] = 0$ for any $X \in \mathcal{D}$ and applying b) of Theorem 6, we have $\widetilde{\nabla}_\xi X = 2\widetilde{T}(\xi, X) = \alpha X$. We also notice that, since $\widetilde{\nabla}\varphi = 0$, then $\widetilde{\nabla}_Z X \in \mathcal{D}$ for any $X \in \mathcal{D}$ and $Z \in \mathfrak{X}(M)$. Hence, for any $X, Y, Z \in \mathcal{D}$, $\widetilde{R}_{XY}Z \in \mathcal{D}$. Supposing $\widetilde{\nabla}\widetilde{R} = 0$, from $(\widetilde{\nabla}_\xi\widetilde{R})(X, Y, Z) = 0$ we get

$\widetilde{R}_{XY}Z = 0$. On the other hand, $\widetilde{R}_{XY}\xi = \widetilde{R}_{\xi X}Y = 0$ for any vector fields X, Y, and thus the curvature tensor $\widetilde{R}$ vanishes. The equivalence of c) and d) is a consequence of Theorem 2. $\square$

We shall discuss now the geometry of CR-integrable almost α-Kenmotsu manifolds such that $h' \neq 0$ and $\widetilde{\nabla}\widetilde{T} = 0$. First of all we prove the following Lemma.

Lemma 1 *Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be a CR-integrable almost α-Kenmotsu manifold. Then the following conditions are equivalent:*

a) $\widetilde{\nabla}\widetilde{T} = 0$,

b) $\widetilde{\nabla}h' = 0$,

c) *the tensor field h' is η-parallel and $\nabla_\xi h' = 0$.*

Proof. Recall that $\widetilde{\nabla}_Z X \in \mathcal{D}$ for any $X \in \mathcal{D}$ and $Z \in \mathfrak{X}(M)$. On the other hand, the torsion $\widetilde{T}$ vanishes on $\mathcal{D}$ and thus $(\widetilde{\nabla}_Z\widetilde{T})(X, Y) = 0$ for any $X, Y \in \mathcal{D}$ and $Z \in \mathfrak{X}(M)$. Now, applying $\widetilde{\nabla}\xi = 0$ and b) of Theorem 6, for any $X \in \mathcal{D}$ and $Z \in \mathfrak{X}(M)$, we have

$$2(\widetilde{\nabla}_Z\widetilde{T})(\xi, X) = \alpha\widetilde{\nabla}_Z(X + h'X) - \alpha(\widetilde{\nabla}_Z X + h'(\widetilde{\nabla}_Z X)) = \alpha(\widetilde{\nabla}_Z h')X,$$

which proves the equivalence of a) and b), since $(\widetilde{\nabla}_Z h')\xi = 0$. Applying (12), we get

$$(\widetilde{\nabla}_Z h')X = \alpha\{(\nabla_Z h')X + \alpha g(h'Z + h'^2 Z, X)\xi\} \in \mathcal{D}$$

and thus, the covariant derivative $\widetilde{\nabla}h'$ vanishes if and only if for any $X, Y \in \mathcal{D}$ and $Z \in \mathfrak{X}(M)$, $g((\nabla_Z h')X, Y) = 0$, which is equivalent to requiring the η-parallelism of the tensor field h' and the vanishing of the covariant derivative $\nabla_\xi h'$. $\square$

Under the hypotheses of the above Lemma, Theorem 4 implies that M^{2n+1} is locally isometric to the warped product (10), where M_0 has dimension 0 or it is a Kähler manifold. For any eigenvalue λ of the operator h', each integral submanifold N of the distribution $[\xi] \oplus [\lambda] \oplus [-\lambda]$ is auto-parallel with respect to the canonical connection $\widetilde{\nabla}$; moreover, the connection induced by $\widetilde{\nabla}$ on N coincides with the canonical connection associated to the induced almost α-Kenmotsu structure.

Let us investigate now the properties of the curvature tensor $\widetilde{R}$ in the non-normal case.

Theorem 7 *Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be a CR-integrable almost α-Kenmotsu manifold such that $h' \neq 0$ and the canonical connection $\widetilde{\nabla}$ has parallel torsion. Then the following conditions are equivalent:*

a) $\widetilde{\nabla}\widetilde{R} = 0$,

b) $\widetilde{R} = 0$,

c) 0 *is a simple eigenvalue of* h' *or the integral submanifolds of the distribution* $[\xi] \oplus [0]$ *have constant Riemannian curvature* $k = -\alpha^2$.

Proof. By Lemma 1, the operator h' is η-parallel and $\nabla_\xi h' = 0$. Hence, applying (11) and (13), we obtain that the curvature tensors R and $\tilde{R}$ are related by

$$\begin{aligned}\widetilde{R}_{XY}Z = R_{XY}Z &- \alpha^2(\eta(Y)g(h'X + h'^2X, Z) - \eta(X)g(h'Y + h'^2Y, Z))\xi \\ &+ \alpha^2\eta(Z)(\eta(Y)(h'X + h'^2X) - \eta(X)(h'Y + h'^2Y)) \\ &+ \alpha^2(g(Y + h'Y, Z)(X + h'X) - g(X + h'X, Z)(Y + h'Y))\end{aligned} \tag{14}$$

for any $X, Y, Z \in \mathfrak{X}(M)$. We know that M^{2n+1} is locally isometric to the warped product (10), where M_0 has dimension 0 or it is a Kähler manifold. Let us consider an eigenvalue $\lambda \neq 0$ of h' and the warped product $B \times_f M_\lambda$ such that $TB = [\lambda]^\perp$, $TM_\lambda = [\lambda]$ and $f = ce^{\alpha(1+\lambda)t}$, $c > 0$. From Proposition 7.42 in [17] it follows that for any $X, Y \in TB$ and $V, W \in [\lambda]$,

$$R_{XY}V = R_{VW}X = 0, \quad R_{VX}Y = -\frac{H^f(X,Y)}{f}V, \quad R_{XV}W = -\frac{g(V,W)}{f}\nabla_X(\mathrm{grad}f).$$

For any vector field Z, we have $Z(f) = \eta(Z)\xi(f) = \alpha(1+\lambda)\eta(Z)f$. Therefore,

$$\begin{aligned}H^f(X,Y) &= X(Yf) - (\nabla_X Y)(f) \\ &= \alpha(1+\lambda)(X(\eta(Y))f + \alpha(1+\lambda)\eta(X)\eta(Y)f - \eta(\nabla_X Y)f) \\ &= \alpha(1+\lambda)((\nabla_X\eta)(Y) + \alpha(1+\lambda)\eta(X)\eta(Y))f \\ &= \alpha^2(1+\lambda)(g(X + h'X, Y) + \lambda\eta(X)\eta(Y))f,\end{aligned}$$

where we used (4). Hence,

$$R_{VX}Y = -\alpha^2(1+\lambda)(g(X + h'X, Y) + \lambda\eta(X)\eta(Y))V.$$

Since $\mathrm{grad}f = \alpha(1+\lambda)f\xi$, then

$$\begin{aligned}\nabla_X(\mathrm{grad}f) &= \alpha(1+\lambda)\{\alpha(1+\lambda)\eta(X)f\xi + \alpha f(X + h'X - \eta(X)\xi)\} \\ &= \alpha^2(1+\lambda)(X + h'X + \lambda\eta(X)\xi)f,\end{aligned}$$

and thus

$$R_{XV}W = -\alpha^2(1+\lambda)g(V,W)(X + h'X + \lambda\eta(X)\xi).$$

Using (14), a straightforward computation shows that

$$\widetilde{R}_{XY}V = \widetilde{R}_{VW}X = \widetilde{R}_{VX}Y = \widetilde{R}_{XV}W = 0. \tag{15}$$

We know that the distribution $[\xi] \oplus [\lambda]$ has totally geodesic leaves with constant Riemannian curvature $\kappa - 2\alpha^2\lambda$, and applying (14) again, we have

$$\widetilde{R}_{UV}W = 0 \tag{16}$$

for any $U, V, W \in [\lambda]$. It remains to analyze the curvatures $\widetilde{R}_{XY}Z$, with $X, Y, Z \in TB$. Considered the eigenvalue $-\lambda$, we regard B as the warped product $B' \times_{f'} M_{-\lambda}$ such that $TB' = [\xi] \oplus \bigoplus_{\mu \neq \pm\lambda}[\mu]$, $TM_{-\lambda} = [-\lambda]$ and $f' = c'e^{\alpha(1-\lambda)t}$, $c' > 0$. Analogous computations give (15) and (16) for any $U, V, W \in [-\lambda]$ and $X, Y \in TB'$. In fact this argument can be applied for each non-vanishing eigenvalue of h', proving that if 0 is a simple eigenvalue, then the curvature tensor $\widetilde{R}$ vanishes on M^{2n+1}.

If 0 has multiplicity greater than 1, we have to analyze the curvature tensor $\widetilde{R}$ on the integral submanifolds of the distribution $[\xi] \oplus [0]$. These leaves are endowed with an almost α-Kenmotsu structure with vanishing operator h' and such that the integral manifolds of $[0]$ are Kähler. Hence, the leaves of $[\xi] \oplus [0]$ are α-Kenmotsu manifolds and, by Proposition 3, the curvature tensor $\widetilde{R}$ vanishes on them if and only if it is parallel with respect to $\widetilde{\nabla}$, or equivalently the leaves have constant Riemannian curvature $k = -\alpha^2$. $\square$

Remark 1 Differently from the normal case, which is described in Proposition 3, in the hypotheses of Theorem 7, conditions a) and b) are not equivalent to the local Riemannian symmetry. Indeed in this case, combining (7) and (11) it follows that the Riemannian curvature satisfies $R_{XY}\xi = 0$ for any $X, Y \in \mathcal{D}$. By Theorem 3 it follows that M^{2n+1} is locally symmetric if and only if $Sp(h') = \{0, 1, -1\}$, with 0 simple eigenvalue.

Corollary 2 *Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be an almost α-Kenmotsu manifold such that $h' \neq 0$ and ξ belongs to the $(\kappa, \mu)'$-nullity distribution, $\mu = -2\alpha^2$. Then $\widetilde{R} = 0$ and the Riemannian curvature tensor is given by*

$$\begin{aligned} R_{XY}Z = {} & \kappa\,\eta(Z)(\eta(Y)X - \eta(X)Y) + \kappa(g(Y,Z)\eta(X) - g(X,Z)\eta(Y))\xi \\ & + \alpha^2(g(Y - h'Y, Z)\eta(X) - g(X - h'X, Z)\eta(Y))\xi \\ & + \alpha^2\eta(Z)(\eta(Y)(X - h'X) - \eta(X)(Y - h'Y)) \\ & - \alpha^2(g(Y + h'Y, Z)(X + h'X) + g(X + h'X, Z)(Y + h'Y)) \end{aligned} \tag{17}$$

for any $X, Y, Z \in \mathfrak{X}(M)$.

Proof. The operator h' is η-parallel and satisfies $\nabla_\xi h' = 0$. The eigenvalue 0 of h' is simple and Theorem 7 implies that the curvature tensor $\widetilde{R}$ vanishes. Moreover, since $h'^2 = \lambda^2(I - \eta \otimes \xi)$, with $\lambda^2 = -1 - \frac{\kappa}{\alpha^2}$, applying (14) we get (17). □

4 The local classification

Theorem 8 *Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ and $(\bar{M}^{2n+1}, \bar{\varphi}, \bar{\xi}, \bar{\eta}, \bar{g})$ be CR-integrable almost α-Kenmotsu manifolds with canonical connections $\widetilde{\nabla}$ and $\widetilde{\bar{\nabla}}$ respectively. Let us suppose that $\widetilde{\nabla}$ and $\widetilde{\bar{\nabla}}$ have parallel torsion and curvature tensors and the operators h' and $\bar{h}'$ associated to the structures have the same eigenvalues with the same multiplicities. Then M^{2n+1} and $\bar{M}^{2n+1}$ are locally equivalent as almost contact metric manifolds.*

Proof. Let us suppose that h' and $\bar{h}'$ have the same eigenvalues with the same multiplicities. Fixed two points $p \in M^{2n+1}$ and $q \in \bar{M}^{2n+1}$, we can choose orthonormal bases $\{\xi_p, e_1, \dots, e_n, \varphi_p e_1, \dots, \varphi_p e_n\}$ of T_pM^{2n+1} and $\{\bar{\xi}_q, \bar{e}_1, \dots, \bar{e}_n, \bar{\varphi}_q \bar{e}_1, \dots, \bar{\varphi}_q \bar{e}_n\}$ of $T_q\bar{M}^{2n+1}$ in such a way that, for any $i = 1, \dots, n$, e_i and $\bar{e}_i$ are eigenvectors of h'_p and $\bar{h}'_q$, respectively, with eigenvalue λ_i, while $\varphi_p e_i$ and $\bar{\varphi}_q \bar{e}_i$ are eigenvectors of h'_p and $\bar{h}'_q$ with eigenvalue $-\lambda_i$. We define a linear isometry $F : T_pM^{2n+1} \to T_q\bar{M}^{2n+1}$ such that

$$F(\xi_p) = \bar{\xi}_q, \qquad F(e_i) = \bar{e}_i, \qquad F(\varphi_p e_i) = \bar{\varphi}_q \bar{e}_i$$

for every $i = 1, \dots, n$. Then, we have

$$F^*\bar{\eta}_q = \eta_p, \qquad F^*\bar{\varphi}_q = \varphi_p, \qquad F^*\bar{h}'_q = h'_p.$$

From Theorem 6, the torsion tensors satisfy $F^*\widetilde{\bar{T}}_q = \widetilde{T}_p$. On the other hand, the curvatures $\widetilde{\bar{R}}$ and $\widetilde{R}$ vanish. It follows that there exists an affine diffeomorphism f of a neighborhood U of p onto a neighborhood V of q such that $f(p) = q$ and the differential of f at p coincides with F [15]. The local diffeomorphism f maps the structure tensors (φ, ξ, η, g) to $(\bar{\varphi}, \bar{\xi}, \bar{\eta}, \bar{g})$, since they are parallel with respect to $\widetilde{\nabla}$ and $\widetilde{\bar{\nabla}}$ respectively. Hence M^{2n+1} and $\bar{M}^{2n+1}$ are locally equivalent as almost contact metric spaces. □

Theorem 9 *Let* $(M^{2n+1},\varphi,\xi,\eta,g)$ *and* $(\bar{M}^{2n+1},\bar{\varphi},\bar{\xi},\bar{\eta},\bar{g})$ *be* CR*-integrable almost* α *and almost* $\bar{\alpha}$*-Kenmotsu manifolds with canonical connections* $\widetilde{\nabla}$ *and* $\bar{\widetilde{\nabla}}$ *respectively. Let us suppose that* $\widetilde{\nabla}$ *and* $\bar{\widetilde{\nabla}}$ *have parallel torsion and curvature tensors. Then* M^{2n+1} *and* $\bar{M}^{2n+1}$ *are locally equivalent as almost contact metric manifolds, up to* $\mathcal{D}$*-homothetic deformations, if and only if the operators* h' *and* $\bar{h}'$ *associated to the structures have the same eigenvalues with the same multiplicities.*

Proof. Let us suppose that h' has eigenvalues $0,\lambda_1,\ldots,\lambda_n,-\lambda_1,\ldots,-\lambda_n$, with $0\leq\lambda_i\leq\lambda_j$ for any $i\leq j$. Analogously, let $0,\bar{\lambda}_1,\ldots,\bar{\lambda}_n,-\bar{\lambda}_1,\ldots,-\bar{\lambda}_n$ be the eigenvalues of $\bar{h}'$, with $0\leq\bar{\lambda}_i\leq\bar{\lambda}_j$ for any $i\leq j$. Since a $\mathcal{D}$-homothetic deformation of the structure leaves the operator h' invariant, if M^{2n+1} and $\bar{M}^{2n+1}$ are locally equivalent up to $\mathcal{D}$-homothetic deformations, then $\lambda_i=\bar{\lambda}_i$ for any $i=1,\ldots,n$. Conversely, let us suppose $\lambda_i=\bar{\lambda}_i$ for any $i=1,\ldots,n$. We apply a $\mathcal{D}$-homothetic deformation with constant $\beta=\frac{\alpha}{\bar{\alpha}}$ to the structure (φ,ξ,η,g), thus obtaining an almost $\bar{\alpha}$-Kenmotsu structure $(\varphi_1,\xi_1,\eta_1,g_1)$ on M^{2n+1} for which the canonical connection has parallel torsion and vanishing curvature and the operator h'_1 has eigenvalues $0,\lambda_1,\ldots,\lambda_n,-\lambda_1,\ldots,-\lambda_n$. From Theorem 8 it follows that $(M^{2n+1},\varphi_1,\xi_1,\eta_1,g_1)$ and $(\bar{M}^{2n+1},\bar{\varphi},\bar{\xi},\bar{\eta},\bar{g})$ are locally equivalent as almost contact metric spaces. □

For any odd dimension $2n+1$ and for any nonnegative and not all vanishing real numbers $\lambda_1,\ldots,\lambda_n$, we give an example of a CR-integrable almost α-Kenmotsu manifold whose canonical connection has parallel torsion and vanishing curvature and such that the operator h' has eigenvalues $0,\lambda_1,\ldots,\lambda_n,-\lambda_1,\ldots,-\lambda_n$. The example is given by a Lie group endowed with a left invariant almost α-Kenmotsu structure.

Let G be the connected and simply connected Lie group of real matrices of the form

$$A=\begin{pmatrix} e^{-\alpha(1+\lambda_1)t} & 0 & \cdots & 0 & 0 & 0 & x_1\\ 0 & e^{-\alpha(1-\lambda_1)t} & \cdots & 0 & 0 & 0 & y_1\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \cdots & e^{-\alpha(1+\lambda_n)t} & 0 & 0 & x_n\\ 0 & 0 & \cdots & 0 & e^{-\alpha(1-\lambda_n)t} & 0 & y_n\\ 0 & 0 & \cdots & 0 & 0 & 1 & t\\ 0 & 0 & \cdots & 0 & 0 & 0 & 1 \end{pmatrix},$$

which is a subgroup of the affine group $Aff(2n+1,\mathbb{R})$ and whose Lie algebra $\mathfrak{g}$ is

given by the real matrices

$$X = \begin{pmatrix} -\alpha(1+\lambda_1)c & 0 & \cdots & 0 & 0 & 0 & a_1 \\ 0 & -\alpha(1-\lambda_1)c & \cdots & 0 & 0 & 0 & b_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -\alpha(1+\lambda_n)c & 0 & 0 & a_n \\ 0 & 0 & \cdots & 0 & -\alpha(1-\lambda_n)c & 0 & b_n \\ 0 & 0 & \cdots & 0 & 0 & 0 & c \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{pmatrix}.$$

For any $i = 1, \dots, n$ denote by X_i and Y_i the matrices in $\mathfrak{g}$ whose coefficients are all vanishing except for $a_i = 1$ and $b_i = 1$ respectively. Let ξ be the matrix corresponding to $c = 1$ and $a_i = b_i = 0$, $i = 1, \dots, n$. Then $\{\xi, X_1, \dots, X_n, Y_1, \dots, Y_n\}$ is a basis of $\mathfrak{g}$ which satisfies

$$[\xi, X_i] = -\alpha(1+\lambda_i)X_i, \qquad [\xi, Y_i] = -\alpha(1-\lambda_i)Y_i,$$
$$[X_i, X_j] = [X_i, Y_j] = [Y_i, X_j] = [Y_i, Y_j] = 0.$$

The above relations imply that G is a solvable non-nilpotent Lie group. We consider the endomorphism $\varphi : \mathfrak{g} \to \mathfrak{g}$ and the 1-form $\eta : \mathfrak{g} \to \mathbb{R}$ such that

$$\varphi(\xi) = 0, \quad \varphi(X_i) = Y_i, \quad \varphi(Y_i) = -X_i, \quad \eta(\xi) = 1, \quad \eta(X_i) = \eta(Y_i) = 0,$$

for any $i = 1, \dots, n$, and denote by g the inner product on $\mathfrak{g}$ such that the basis $\{\xi, X_i, Y_i\}$ is orthonormal. The tensors defined on $\mathfrak{g}$ determine a left invariant CR-integrable almost α-Kenmotsu structure (φ, ξ, η, g) on G. Each X_i is an eigenvector of h' with eigenvalue λ_i, while each Y_i is eigenvector with eigenvalue $-\lambda_i$. Moreover, the canonical connection $\widetilde{\nabla}$ coincides with the left invariant connection on the Lie group, which has vanishing curvature and parallel torsion. Indeed, denoting by ∇' the left invariant connection on G, the structure tensor fields φ, ξ, η, g are parallel with respect to ∇'. Since the torsion T' is given by $2T'(X,Y) = -[X,Y]$ for any $X, Y \in \mathfrak{g}$, T' satisfies a), b), c) of Theorem 6, so that ∇' coincides with the canonical connection associated to the structure (φ, ξ, η, g).

Finally, considering the coordinate system $\{t, x_1, y_1, \dots, x_n, y_n\}$ on the Lie group G, we have

$$X_i = e^{-\alpha(1+\lambda_i)t}\frac{\partial}{\partial x_i}, \qquad Y_i = e^{-\alpha(1-\lambda_i)t}\frac{\partial}{\partial y_i},$$

$$\xi = \frac{\partial}{\partial t}, \qquad \eta = dt, \qquad \varphi = \sum_{i=1}^{n} e^{2\alpha\lambda_i t} dx_i \otimes \frac{\partial}{\partial y_i} - \sum_{i=1}^{n} e^{-2\alpha\lambda_i t} dy_i \otimes \frac{\partial}{\partial x_i}, \qquad (18)$$

$$g = dt \otimes dt + \sum_{i=1}^{n} e^{2\alpha(1+\lambda_i)t} dx_i \otimes dx_i + \sum_{i=1}^{n} e^{2\alpha(1-\lambda_i)t} dy_i \otimes dy_i. \tag{19}$$

Therefore, from Theorem 8 we have the following

Proposition 4 *Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be an almost α-Kenmotsu manifold. Then M^{2n+1} is a CR-integrable almost α-Kenmotsu manifold with canonical connection $\widetilde{\nabla}$ satisfying $\widetilde{\nabla}\widetilde{T} = 0$ and $\widetilde{\nabla}\widetilde{R} = 0$ if and only if for any point $p \in M^{2n+1}$ there exists an open neighbourhood with local coordinates $\{t, x_1, y_1, \ldots, x_n, y_n\}$ on which* (18) *and* (19) *hold.*

As a consequence of Theorem 9, we can associate to each almost α-Kenmotsu $(\kappa, \mu)'$-space M^{2n+1}, with $h' \neq 0$, the real number

$$I_{M^{2n+1}} = \frac{\kappa}{\alpha^2},$$

which classifies such spaces up to $\mathcal{D}$-homothetic deformations, as stated in the following Theorem.

Theorem 10 *Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be an almost α-Kenmotsu manifold with $h' \neq 0$ and ξ belonging to the $(\kappa, \mu)'$-nullity distribution, $\mu = 2\alpha^2$, and let $(\bar{M}^{2n+1}, \bar{\varphi}, \bar{\xi}, \bar{\eta}, \bar{g})$ be an almost $\bar{\alpha}$-Kenmotsu manifold with $\bar{h}' \neq 0$ and $\bar{\xi}$ belonging to the $(\bar{\kappa}, \bar{\mu})'$-nullity distribution, $\bar{\mu} = -2\bar{\alpha}^2$. Then, M^{2n+1} and $\bar{M}^{2n+1}$ are locally equivalent up to a $\mathcal{D}$-homothetic deformation, as almost contact metric spaces, if and only if $I_{M^{2n+1}} = I_{\bar{M}^{2n+1}}$.*

Proof. The operators h' and $\bar{h}'$ have eigenvalues $0, \lambda, -\lambda$ and $0, \bar{\lambda}, -\bar{\lambda}$ respectively, where 0 is simple, $\lambda = \sqrt{-1 - \frac{\kappa}{\alpha^2}}$ and $\bar{\lambda} = \sqrt{-1 - \frac{\bar{\kappa}}{\bar{\alpha}^2}}$. The result immediately follows from Theorem 9. In particular, if $I_{M^{2n+1}} = I_{\bar{M}^{2n+1}}$, we have to apply a $\mathcal{D}$-homothetic deformation to the structure (φ, ξ, η, g) with constant $\beta = \frac{\alpha}{\bar{\alpha}} = \sqrt{\frac{\kappa}{\bar{\kappa}}}$. □

Notice that $I_{M^{2n+1}} < -1$ since $\kappa < -\alpha^2$. By Corollary 1, M^{2n+1} is locally symmetric if and only if $I_{M^{2n+1}} = -2$. For any dimension $2n+1$ and for any value of the invariant $I < -1$, an explicit example of these manifolds is given by the Lie group G described above, with $\lambda_1 = \ldots = \lambda_n = \lambda$. Indeed, considered the left invariant almost α-Kenmotsu structure (φ, ξ, η, g), by Proposition 2, the characteristic vector field ξ belongs $(\kappa, \mu)'$-nullity distribution, with $\kappa = -\alpha^2(1+\lambda^2)$ and $\mu = -2\alpha^2$. The invariant is

$$I_G = -1 - \lambda^2$$

which attains any real value smaller than -1.

Remark 2 In [4] E. Boeckx introduces a scalar invariant which classifies, up to $\mathcal{D}$-homothetic deformations, non-Sasakian contact metric manifolds whose characteristic vector field belongs to a (κ, μ)-nullity distribution. The proof of the equivalence theorem involves the Levi-Civita connection and the properties of the covariant derivatives of the Riemannian curvature R and the structure tensors φ, ξ, η, g. Here, the classification theorem for almost α-Kenmotsu $(\kappa, \mu)'$-spaces, is obtained as a consequence of the more general Theorem 9, which involves the canonical connection and the parallelism with respect to it of the torsion $\widetilde{T}$, the curvature $\widetilde{R}$ and the structure tensors. One could wonder if it is possible to prove the equivalence theorem for non-Sasakian (κ, μ)-contact metric spaces by using the Tanaka-Webster connection. The answer is negative, since in this case the torsion tensor $\widetilde{T}$ and the curvature tensor $\widetilde{R}$ of the Tanaka-Webster connection $\widetilde{\nabla}$ in general are not parallel with respect to $\widetilde{\nabla}$; this happens if and only if $\mu = 2$, as proved in [5].

We conclude the analysis of the local geometry of almost α-Kenmotsu $(\kappa, \mu)'$-spaces with the following result.

Proposition 5 *Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be an almost α-Kenmotsu manifold such that $h' \neq 0$ and ξ belongs to the $(\kappa, \mu)'$-nullity distribution, $\mu = -2\alpha^2$. Let g' be the Riemannian metric locally defined by the $\mathcal{D}$-conformal change*

$$g' = e^{-2\alpha t} g + (1 - e^{-2\alpha t}) \eta \otimes \eta.$$

Then (φ, ξ, η, g') is an almost cosymplectic structure such that ξ belongs to the κ_c-nullity distribution, with $\kappa_c = \kappa + \alpha^2$.

Proof. The fundamental 2-form Φ' associated to the structure (φ, ξ, η, g') is locally given by $\Phi' = e^{-2\alpha t}\Phi$ and thus $d\Phi' = 0$, so that (φ, ξ, η, g') is an almost cosymplectic structure. By Proposition 4, for any point $p \in M^{2n+1}$ there exist local coordinates $\{t, x_1, y_1, \ldots, x_n, y_n\}$, such that

$$\xi = \frac{\partial}{\partial t}, \qquad \eta = dt, \qquad \varphi = e^{2\alpha\lambda t} \sum_{i=1}^{n} dx_i \otimes \frac{\partial}{\partial y_i} - e^{-2\alpha\lambda t} \sum_{i=1}^{n} dy_i \otimes \frac{\partial}{\partial x_i},$$

$$g = dt \otimes dt + e^{2\alpha(1+\lambda)t} \sum_{i=1}^{n} dx_i \otimes dx_i + e^{2\alpha(1-\lambda)t} \sum_{i=1}^{n} dy_i \otimes dy_i,$$

with $\lambda = \sqrt{-1 - \frac{\kappa}{\alpha^2}}$. Hence, the Riemannian metric g' is locally given by

$$g' = dt \otimes dt + e^{2\alpha\lambda t} \sum_{i=1}^{n} dx_i \otimes dx_i + e^{-2\alpha\lambda t} \sum_{i=1}^{n} dy_i \otimes dy_i.$$

By a result of P. Dacko [6], it follows that (φ, ξ, η, g') is an almost cosymplectic structure such that ξ belongs to the κ_c-nullity distribution, with $\kappa_c = -\lambda^2\alpha^2 = \kappa + \alpha^2 < 0$. □

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